

## **Mathematical Programming with Equilibrium Constraints – Expanding the Scope of Optimization**

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### **Summary**

*What have electricity markets, toll-road pricing, and quasi-brittle fracture identification in common? They can all be modeled as optimization problems with equilibrium constraints. Researchers at Argonne National Laboratory have developed new theoretical insight and computational techniques to tackle this growing class of challenging problems.*

Optimization, the art of finding the minimum value of an objective function subject to constraints, is one of the fundamental modeling paradigms of mathematics and computational science. It appears in virtually any area of science and engineering. Yet in spite of decades of advances in the theory and algorithms for optimization, important challenges remain, especially in the case where both the function and constraints are nonlinear, giving rise to the nonlinear optimization case. Overcoming these challenges is an important step toward creating robust software for many scientific computation applications.

Recently, an important class of problems that poses special challenges has been identified by the optimization community: mathematical programs with equilibrium constraints (MPECs). MPECs arise, for example, in the modeling of electricity markets as so-called Stackelberg games. These problems are characterized by a dominant producer (the leader) that can exercise market power by anticipating the

optimal responses of the smaller producers (the followers). The leader maximizes its profits subject to the optimal response of the followers. The result is an equilibrium constraint in the leader's optimization problem and, thus, an MPEC.

MPECs are computationally challenging problems because they violate standard stability assumptions. To make matters worse, early numerical experience was so disappointing that researchers concluded that classical nonlinear optimization algorithms would never solve MPECs reliably. This conclusion was unfortunate, since the set of applications described by MPECs is vast, including many diverse applications such as fracture identification, optimal control of cost-efficient robots, optimal design of contact configurations in mechanical and civil engineering, multiscale modeling in nanoscience, and data classification.

Recently, our group has shown that this gloomy prognosis was premature. By combining new theoretical insight and novel

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computational developments, we have been able to reliably and efficiently solve a large class of MPECs. Our techniques routinely solve applications that are orders of magnitude larger and more complex than previously possible, extending the scope of this important new computational paradigm. An example of the solution of a medium-scale problem is presented in Figure 1, where the optimal contact region between an elastic membrane and a rigid obstacle, defined by an MPEC, is successfully solved by our methods.

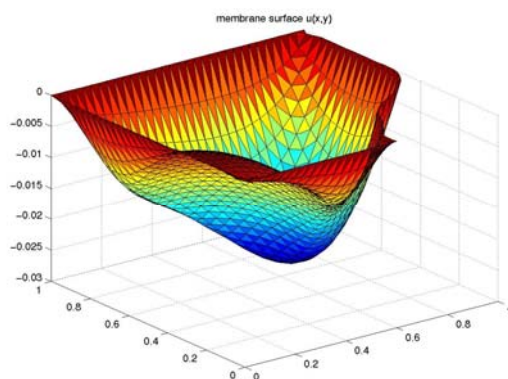


Figure 1: Minimal surface membrane over an obstacle

The equilibrium constraint models the contact area, where the membrane rests on the obstacle. This contact area typically has a complex geometric structure, shown in red in Figure 2.

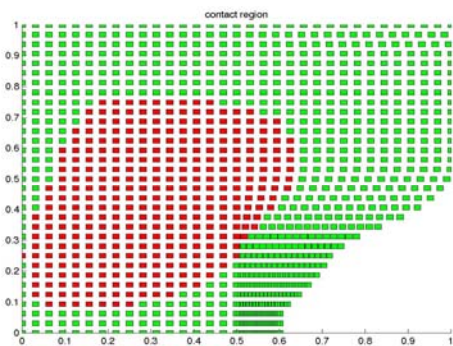


Figure 2: Obstacle contact area in red

Many popular nonlinear optimization codes have been modified to include our techniques and to expand their set of problems that can be reliably solved. Another indication of the increasing impact of our work is the fact that our five research papers have collected more than 100 citations (even though four have not yet appeared in the literature!).

In addition, we have collected over 150 MPECs in a test problem library, [www.mcs.anl.gov/~leyffer/MacMPEC/](http://www.mcs.anl.gov/~leyffer/MacMPEC/), which has become the de facto benchmark for research on equilibrium constraints.

As a result of our work, we are now helping policy makers to better understand how electricity markets operate. Moreover, engineers are able to better predict how cracks in dams and other brittle structures develop.

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